MathsBites by Clifford the Dog

Nested Radicals and Other Recurrences

Nested radicals

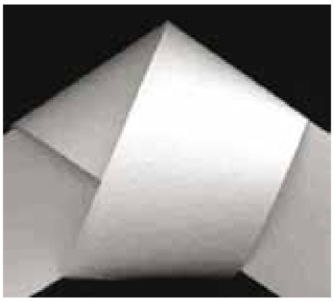
If a regular pentagon is constructed so that its vertices touch the sides of a unit circle (https://en.wikipedia.org/wiki/Pentagon) then its side length is

$$\sqrt{\frac{5-\sqrt{5}}{2}}.$$

This number is an example of a *nested radical* (http://en.wikipedia.org/wiki/Nested_radical). Another example is the *infinite* nested radical

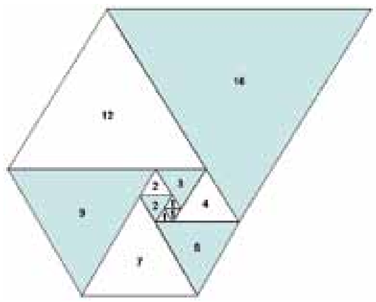
$$x=\sqrt{2+\sqrt{2+\sqrt{2+\ldots}}}.$$

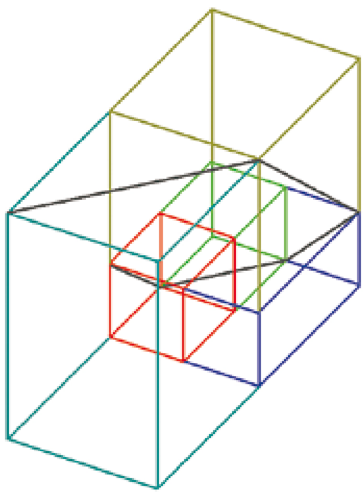
Clearly x is positive, and noting that x recurs as the second term within the radical, the above expression is the same as $x=\sqrt{2+x}$. A bit of algebra results in a quadratic equation from which one can obtain $x = 2$.



Activity: Generalise the above to show that $\sqrt{n+\sqrt{n+\sqrt{n+\ldots}}}=\frac{1+\sqrt{1+4n}}{2}$ for any natural number n.

The Padovan sequence and the plastic number





The Padovan sequence (http://en.wikipedia.org/wiki/Padovan_sequence) is a bit like the Fibonacci sequence. It is specified by the initial values $p(1) = p(2) = p(3) = 1$, and the recurrence relation:

$p(n) = p(n-2) + p(n-3)$. So $p(4) = p(2) + p(1) = 1 + 1 = 2$.

Find the first 20 terms of the Padovan sequence. This sequence can be represented in two dimensions using equilateral triangles. As with Fibonacci sequence, that ratio of successive terms

$$\frac{p(n+1)}{p(n)}$$

has a limiting value as n tends to infinity. This is called the *plastic number p*. Show that $p \approx 1.324718$.

It can be expressed as an infinite nested radical involving cube roots where

$$p=\sqrt[3]{1+\sqrt[3]{1+\sqrt[3]{1+\ldots}}}\ .$$

The Padovan sequence can also be represented in three dimensions using ratios of cuboid volumes (http://wayback.archive.org/web/20120320051231/http:/members.fortunecity.com/templarser/padovan.html)

Activity: Show that p is the unique real solution to the equation $p^3 - p - 1 = 0$.