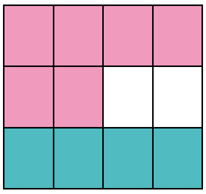
MathsBites by Clifford the Dog

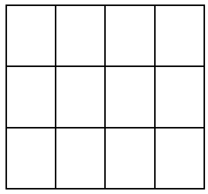
Some unit fraction sums

Sums of consecutive unit fractions

Unit fractions are fractions of the form $\frac{1}{2}, \frac{1}{3}, \frac{1}{4} \ldots$ and so on. Suppose the rectangle shown has unit area. Explain how the shaded areas can be used to illustrate $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$.



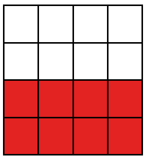
Shade the rectangle on the right to illustrate $\frac{1}{3} + \frac{1}{4} = \frac{7}{12}$.



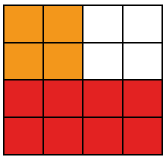
Calculate $\frac{1}{4} + \frac{1}{5}; \frac{1}{5} + \frac{1}{6}; \frac{1}{6} + \frac{1}{7}$ and so on. Find a rule for the sum of two consecutive unit fractions, and explain why it works.

Almost filling up a square

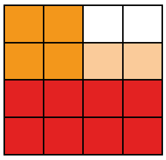
Consider the sequence of large squares of unit area which are progressively shaded in as shown:



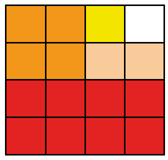
$\frac{1}{2}$



$\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$



$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$

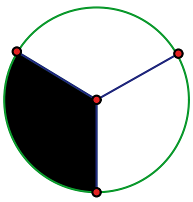


$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}$

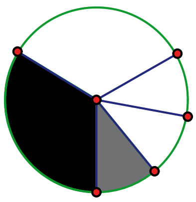
If this pattern of shading in is continued indefinitely, the large square of unit area will never be completely filled in, however one can fill it in almost as close to completely as desired. At any stage of n 'shading ins' the unfilled area is $\left(\frac{1}{2}\right)^n = \frac{1}{2^n}$ and this becomes increasingly small and tends to zero as n increases. The shaded area is thus $1 - \frac{1}{2^n}$ and tends to 1 as n increases. Alternatively, calculate the sum to infinity of the infinite geometric series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$.

Almost filling up half a pie

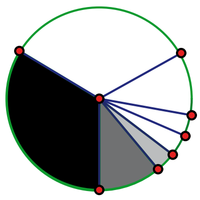
Consider a sequence of unit circles (radius = 1 and area = π) progressively shaded in as shown:



One third of the circle



One third of the circle and one third of a remaining third



One third of the circle, one third of a remaining third, and one third of one third of a remaining third

Show that continuing this process indefinitely results in almost half of the circle being filled.

More infinite sums

Consider the family of infinite geometric series of the kind: $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$; $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{181} + \ldots$; $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \ldots$; and so on. These each have first term and common ratio equal and specified by a unit fraction. Find a general result for the corresponding sums to infinity and explain why it works.