MathsBites by Clifford the Dog

The Fibonacci sequence and matrices

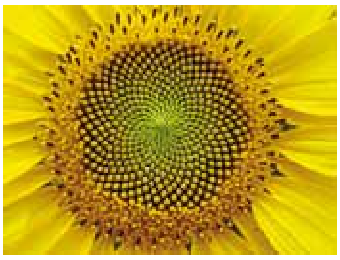
The Fibonacci sequence

The Fibonacci sequence {1, 1, 2, 3, 5, 8 …} is well known for historical interest and also for application to contexts such as rabbit populations and patterns in nature, for example: www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/fibnat.html and http://www.etereaestudios.com/docs_html/nbyn_htm/nbyn_mov_youtube.htm

The Fibonacci sequence is readily recognisable as a recurrence relation, where after the initial terms any subsequent term is the sum of the two previous terms:

$$f(1) = 1,\ f(2) = 1,\ f(n) = f(n-1) + f(n-2)$$

So $f(3) = f(2) + f(1) = 1 + 1 = 2$; $f(4) = f(3) + f(2) = 2 + 1 = 3$ … . Use this relation to calculate the first 20 terms of the sequence. This can be generalised to other similar sequences with different initial terms.



Illustrations by Jen Maddocks Designs of www.jencropable.com

Activity: Find an explicit rule for $f(n)$: see, for example http://en.wikipedia.org/wiki/Fibonacci_number

Matrices and Fibonacci sub-sequences

Leonardo Fibonacci (~1170 – 1250)

http://en.wikipedia.org/wiki/File:Fibonacci.jpg

Matrices can be used to generate Fibonacci sub-sequences of the form $\{f(n-1), f(n), f(n+1)\}$ for a given value of n. If **F** is the matrix

$$\mathbf{F} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

Then the elements $\{f_{22}, f_{12} = f_{21}, f_{11}\}$ of the matrix $\mathbf{F}^n$ generate the sub-sequence $\{f(n-1), f(n), f(n+1)\}$. For example,

$$\mathbf{F}^6 = \begin{pmatrix} 13 & 8 \\ 8 & 5 \end{pmatrix}$$

This generates the sub-subsequence

$$\{f_{22}, f_{12} = f_{21}, f_{11}\} = \{f(5), f(6), f(7)\} = \{5, 8, 13\}$$

Activity: Use mathematical induction to prove that

$$\mathbf{F}^n = \begin{pmatrix} f(n+1) & f(n) \\ f(n) & f(n-1) \end{pmatrix}$$

where $f(n)$ is the nth term of the Fibonacci sequence. *Hint:* Assign $f(0) = 0$ then $F = F^1$ satisfies the initial condition and the inductive step follows from computation of $\mathbf{F}^{n+1} = \mathbf{F}^n \times \mathbf{F}$.